On Special Geometry of Generalized G Structures and Flux Compactifications

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Dreams of A. Einstein: Unifications of interacting forces of nature

1920's known forces: Gravity and Electro-Magnetic forces(Matters)

General Relativity: Einstein-Hilbert action

$$S(g) = \int_{M^4} R\sqrt{-g}d^4x$$

$$\delta S = 0$$

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 0.$$

Electro-Magnetic forces obey Maxwell equations:

$$S = \int -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{4} J_\mu A_\mu$$

$$F = dA = \Sigma F_{\mu\nu} dx^\mu \wedge dx^\nu, F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

$$dF = 0, *d(*F) = J.$$

Unification of gravity and electro-magnetic fields: Kaluza-Klein compactifications

$$M^5 = M^4 \times S^1$$

$$ds^2 = \Sigma g_{mn} dx^m dx^n$$

$$= e^{-2\Phi/3} \Sigma g_{\mu\nu} dx^\mu dx^\nu + e^{4\Phi/3} (d\theta + A_\mu dx^\mu)^2$$

$$g_{\mu\nu}(x, \theta) = \Sigma g_{\mu\nu;n} e^{2\pi n\theta}$$

$$A_\mu(x, \theta) = \Sigma A_{\mu;n} e^{2\pi n\theta}$$

Only keeping the lightest modes: $g_{\mu\nu;0}, A_{\mu;0}$

The Einstein-Hilbert action

$$S(g) = \int_{M^5} R\sqrt{-g}d^5x$$

reduces to the Einstein-Hilbert-Yang-Mills action

$$S(g, A, \Phi) = \int_{M^4} \sqrt{-g}d^4x(R + |d\Phi|^2 - \frac{1}{2}F_{\mu\nu}F^{\mu\nu})$$

Conclusion:

We have unified gravity-electro-magnetic forces by using a five dimensional gravity!

String/M Theory is to realize A. Einstein's dream in broadest generality

We have four fundamental forces: gravity and matters: electro-magnetic, weak and strong

Matters obey laws in gauge theories with gauge groups: $U(1), SU(2) \times U(1), SU(3) \times SU(2) \times U(1)$

Standard models were constructed in 1970's and it fits with experiments very well

Gravity is generalized to super-gravity to incorporate super-symmetry

SUGRA is a local gauge theory with gauge symmetry the SUSY algebra

M Theory: 11 dimensional Super-gravity

Fields: a metric G_{MN}, a gravitino ψ_M, and a three form A_{MNP}

The bosonic action of the 11D Supergravity:

$$2kS = \int d^{11}x\sqrt{-G}(R - \frac{1}{2}|F_4|^2) - \frac{1}{6}\int A_3 \wedge F_4 \wedge F_4.$$

Supersymmetric solutions:

$$\delta\Psi_M = \nabla_M \epsilon + \frac{1}{12}(\Gamma_M F^4 - 3F^4_M)\epsilon = 0,$$

ϵ are Killing spinors.

Type IIA String theory from M theory by dimensional reduction

$$M^{11} = M^{10} \times S^1$$

$$ds^2 = \Sigma g_{mn} dx^m dx^n$$

$$= e^{-2\Phi/3} \Sigma g_{\mu\nu} dx^\mu dx^\nu + e^{4\Phi/3} (d\theta + A_\mu dx^\mu)^2$$

$$A^{11}_{\mu\nu\rho} = A_{\mu\nu\rho}, A^{11}_{\mu\nu 11} = B_{\mu\nu}$$

The bosonic action of Type IIA is:

$$S = S_{NS} + S_R + S_{CS}$$

$$S_{NS} = \frac{1}{2k^2} \int d^{10}x \sqrt{-g} (R + 4\partial_\mu \Phi \partial^\mu \Phi - \frac{1}{2}|H_3|^2),$$

$$S_R = -\frac{1}{4k^2} \int d^{10}x \sqrt{-g} (|F_2|^2 + |F_4|^2),$$

$$S_{CS} = -\frac{1}{k^2} \int B_2 \wedge A_4 \wedge A_4.$$

Super-symmetric solutions

$$\delta\Psi_\mu = (\nabla_\mu - \frac{1}{4}H^3_\mu\Gamma_{11} - \frac{1}{8}e^\Phi F_{\nu\rho}\Gamma^{\nu\rho}_\mu\Gamma_1 + \frac{1}{8}e^\Phi F^4\Gamma_\mu)\epsilon = 0,$$

$$\delta\lambda = (-\frac{1}{3}\Gamma^\mu\partial_\mu\Phi\Gamma_{11} + \frac{1}{6}H^3 - \frac{1}{4}e^\Phi F^2 + \frac{1}{12}e^\Phi F^4\Gamma_{11})\epsilon = 0,$$

$$dH = 0.$$

Compactifications of 10D theory to a 4D theory

$$M^{1,9} = R^{1,3} \times M^6$$

$$\epsilon_{1,2} = \Sigma \xi_{1,2} \otimes \eta_{1,2}$$

No fluxes: Covariant constant spinor and Calabi-Yau manifold

$$\nabla \eta = 0$$

Yau's theorem: For a Kahler manifold M with $c_1(TM) = 0$, there exists a Ricci flat metric.

Moduli space of Calabi-Yau manifolds

Symplectic structures ω and complex structures Ω

$$\mathcal{M} = \mathcal{M}_K \times \mathcal{M}_C \subset H^{1,1}(M) \times H^{2,1}(M)$$

Special geometry over $\mathcal{M}$

$\mathcal{M}$ is a Kahler manifold with the Kahler metric:

$$ds^2 = \frac{1}{V}\int_{M^6} g^{a\bar{b}} g^{c\bar{d}} (\delta g_{ac}\delta g_{\bar{b}\bar{d}} + (\delta g_{a\bar{d}}\delta g_{c\bar{b}} - \delta B_{a\bar{d}}\delta B_{c\bar{b}}) d^6 x.$$

It is a Kahler metric with Kahler potential:

$$e^{K^{2,1}} = -i\int \Omega \wedge \bar{\Omega},$$

$$e^{K^{1,1}} = -i\int \omega \wedge \omega \wedge \omega.$$

Much of developments in string theory depend on this special geometry!

Mirror symmetry: one Calabi-Yau M_1 is mirror to another Calabi-Yau M_2

$$\mathcal{M}_K(M_1) = \mathcal{M}_C(M_2).$$

Web of dualities of string theories

Turning on fluxes: Nervu-Schwarz fluxes and Ramond-Ramond fluxes

Super-symmetric solutions

$$\delta\Psi_\mu = (\nabla_\mu - \frac{1}{4}H^3_\mu\Gamma_{11} - \frac{1}{8}e^\Phi F_{\nu\rho}\Gamma^{\nu\rho}_\mu\Gamma_1 + \frac{1}{8}e^\Phi F^4\Gamma_\mu)\epsilon = 0,$$

$$\delta\lambda = (-\frac{1}{3}\Gamma^\mu\partial_\mu\Phi\Gamma_{11} + \frac{1}{6}H^3 - \frac{1}{4}e^\Phi F^2 + \frac{1}{12}e^\Phi F^4\Gamma_{11})\epsilon = 0,$$

$$dH = 0.$$

Existence of two non-vanishing spinors: an almost generalized $SU(3) \times SU(3)$ structure

Structure group reduction from $SO(6,6)$ to $SU(3,3)$

We have two $SU(3)$ invariant spinors $\epsilon_{1,2}$

$$\rho = \Sigma \bar{\epsilon_2} \Gamma^{\mu_1 \dots \mu_p} \epsilon_1 dx_{\mu_1} \wedge \dots \wedge dx_{\mu_p}$$

$$\epsilon_1 = \epsilon_2, \rho = 1 + \omega^{1,1} + \Omega^{3,0} + \text{Hodge dual}$$

Supersymmetric solutions are integrable generalized $SU(3) \times SU(3)$ structures

$$d_H(\rho) = 0,$$

$$d_H \hat{\rho} = *F,$$

$$d_H = d + H \wedge .$$

Those are generalized Maxwell-Hodge equations.

The moduli space of almost generalized $SU(3) \times SU(3)$ structures over a vector space:

$$\mathcal{M} = SO(6,6)/SU(3,3) = U_\rho/\mathbf{C}^*$$

Over a manifold we have a bundle:

$$U_\rho/\mathbf{C}^* \to \mathcal{E} \to M^6$$

The space of generalized $SU(3) \times SU(3)$ structures is:

$$\bar{\mathcal{M}} = \{\Phi \in \mathcal{E}(M^6) | d\Phi = 0\}$$

Turning on all fluxes: the space of $\mathcal{N} = 1$ string vacua is:

$$\bar{\mathcal{M}} = \{\Phi \in \mathcal{E}(M^6) | d_H \Phi = 0, d_H \hat{\rho} = *F\}$$

Finally, the moduli space of N=1 string vacua is:

$$\mathcal{M} = \bar{\mathcal{M}} / D\bar{iff_0}(M^6)$$

The Period map: Taking d_H cohomology classes

Gualtieri: the $\partial_H\bar{\partial}_H$ lemma is true for a generalized Kahler manifold

Goto: If the $\partial_H\bar{\partial}_H$ lemma is true, then the period map is injective.

$$\bar{\mathcal{M}} = \{\Phi \in \mathcal{E}(M^6) | d_H\Phi = 0, d_H\hat{\rho} = 0\}$$

is included in the space of d_H cohomologies

Special geometries over $\mathcal{M}$

Constant symplectic structure over $\mathcal{M}$

$$\omega(\Phi_1, \Phi_2) = \int_{M^6} < \Phi_1, \Phi_2 >,$$

Mukai Pairing $:< \Phi_1, \Phi_2 >= \Sigma_p \Phi_{1,p} \wedge \Phi_{2,n-p}$.

We have: $d\omega = 0$,

$$\omega = \Sigma dx^K \wedge dy_K.$$

Complex structure over $\mathcal{M}$

Stable spinor decomposed into pure spinors:$\Phi = \phi + \hat{\phi}$

$$X : \rho = \phi + \hat{\phi} \rightarrow \hat{\rho} = -i\phi + i\bar{\phi},$$

$$DX.DX = -Id.$$

$I = DX$defines an intregrable complex structure over$\mathcal{M}$

$$\omega(\Phi, \bar{\Phi}) = (\bar{Z}^K F_K - Z^K \bar{F}_K),$$

Z^K, F_K are two independent complex coordinates.

Hitchin functional and the Kahler metric over $\mathcal{M}$

$$H(\Phi) = \int_{M^6} -i < \phi, \bar{\phi} > .$$

The Kahler metric over $\mathcal{M}$ is:

$$ds^2 = \Sigma \partial_{\alpha\beta} H d\chi^\alpha \otimes d\chi^\beta.$$

The Kahler potential is:$K = -\log H$.

We finally have:

$$e^{-K(\Phi)} = H(\Phi) = i\omega(\Phi, \bar{\Phi}) = i(\bar{Z}^K F_K - Z^K \bar{F}_K).$$

Topological strings

The moduli space is acted by mapping class groups, $Diff^+(M)/Diff_0^+(M) \times H_2(M, \mathbf{Z})$

The mapping class group acts on the space of cohomologies which is a vector space

This gives a flat connection on the moduli space of generalized Calabi-Yau manifold (Gauss-Manin connection)

We can construct topological strings over a generalized Calabi-Yau manifold.

Q: Can we generalize Ooguri-Strominger-Vafa's conjecture to generalized Calabi-Yau manifolds?

Flux compactifications and Supersymmetry breaking

By introducing RR fluxes, we break half of supersymmetries

$$d_H \rho = 0,$$

$$d_H \hat{\rho} = *F.$$

RR fluxes are coupled with D-branes, generalized D-branes are generalized sub-manifolds

Q: How about the moduli space and what are the BCFT?

Q: Can we build more realistic standard model with all fluxes turning on?

Mirror symmetry for generalized Calabi-Yau manifolds

It is just exchanging two generalized complex structures

In the case of Calabi-Yau, it is exchanging complex and symplectic structures

Q: How about compactifications of M theory to a seven dimensional manifolds?

The answer would be generalized G_2 manifolds.

Special geometries for generalized G_2 structures

Still have stable spinors and constant symplectic forms and Hitchin functional

Q: Are they giving the same set of 4D N=1 string vacua?

A: They should be the same and this is duality!